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Stability of Large Space Structure Control Systems Using Positivity Concepts

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A robust stability test and associated design procedure based on the positivity of operators is proposed. The test does not rely on modal truncation or high-order truth models of the structure and is independent of the numerical values of the modal data. The stability criterion is applied to the plant (structure) and the controller individually, assuring global stability when the loop is closed by negative feedback. Therefore, design/stability evaluations need only iterate on the low-order controller part of the loop. The method can be extended to nonlinear systems.

Introduction

MODAL truncation in the dynamic models of large space structures (LSS) can cause well-known stability problems[1-4] when these models are used as the design basis for active structure control. Therefore, most active structure control methods proposed so far demand an extensive stability evaluation after completion of the control system design. This is done by using very high-order "truth models" of the structure, accounting for as many residual modes (modes previously neglected in the controller design) as is computationally practical. When the system can justifiably be assumed linear, solution of a large eigenvalue problem will suffice to establish stability; however, if the system contains significant nonlinearities in the actuators and sensors, extensive simulations of a very high-order system would be required. Aside from the fact that in either case such stability evaluations are time-consuming and costly, they are usually based on erroneous mathematical models of the structure, which casts doubt on the validity of the results obtained. It is well known from on-orbit experience (e.g., Skylab) that modal models of large structures obtained by finite-element methods (e.g., NASTRAN) must be expected to contain errors greater than 10% in the determined modal frequencies and mode shapes. An additional source of error is that even in a high-order truth model, modal truncation must occur at some place.

For the above reasons there exists an urgent need for the following.

1) Robust stability tests independent of the number and exact numerical values of modal frequencies and mode shapes, and tests that can be extended to nonlinear (and potentially even time-varying) systems.

2) Control laws which can be assured to remain stable while a system is undergoing preoperational on-orbit tests to identify system parameters for subsequent controller tuning and performance improvement.

3) Design procedures which can assure stability a priori, or at least require iteration of the design/stability evaluations on no more than the low-order controller part of the loop.

It is clear that for this purpose design methods and stability tests broader than "poles in the left-half plane" are required. We are proposing such a method based on the positivity of operators. Physically, the method has the interpretation of assuring that the closed-loop system will be energy dissipative.

The proposed method makes extensive use of functional analysis techniques and is an outgrowth of Popov's[5,6] work on absolute and hyperstability, which was subsequently extended by many researchers (for example, Refs. 7-12). The basic underlying theory is general and *not* restricted to linear and time-invariant plants. In fact, the method has in the past been associated with the stability of nonlinear systems (Lur'e problem[13]) where both the nonlinear and linear parts of the system must meet certain positivity conditions. These conditions are only sufficient conditions for stability and therefore tend to be conservative at times. Since well-known necessary and sufficient conditions exist to establish the stability of linear systems, it is apparent why positivity techniques have generally not been applied to linear systems in the past.

The stability problem in the control of large space structures (LSS) differs from more conventional linear systems, because the plant to be controlled is theoretically of infinite order (and practically of very high order), while the controller/estimator must, of necessity, be of relatively low order. It appears desirable, therefore, to use tests which do not require explicit evaluation of the closed-loop system, but which impose conditions individually on the plant and the controller, assuring stability when the loop is closed by negative feedback. The stability theory based on the positivity of operators is a powerful method having just this characteristic.

Preliminaries

Before proceeding to the main results applicable to LSS controllers, we need to establish some definitions[14,15] and notation.

Definition 1: A square transfer matrix $Z(s)$ is called "positive real" if

1) $Z(s)$ has real elements for real s.

2) $Z(s)$ has elements which are analytic for $\mathrm{Re}[s]>0$.

3) $Z^*(s)+Z(s)$ is nonnegative definite for $\mathrm{Re}[s]>0$, where * denotes the complex conjugate transpose.

Definition 2: A square transfer matrix $Z(s)$ is "strictly positive real" if

1) $Z(s)$ has real elements for real s.

2) $Z(s)$ has elements which are analytic for $\mathrm{Re}[s]\geq 0$.

3) $Z^*(j\omega)+Z(j\omega)$ is positive definite for all real ω.

Remark 1: If the transfer matrix describing a system is strictly positive real, then this implies that the system is energy dissipating. If the system is an electric network, then it is realizable by passive circuit components.

Assume that $Z(s)$ is a matrix of rational functions in s with $Z(\infty)=0$. Then a matrix triple $\{A, B, C\}$ is termed a

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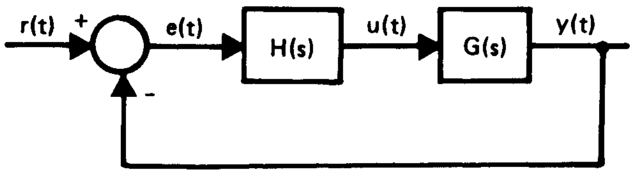

Fig. 1 System S (vector valued).

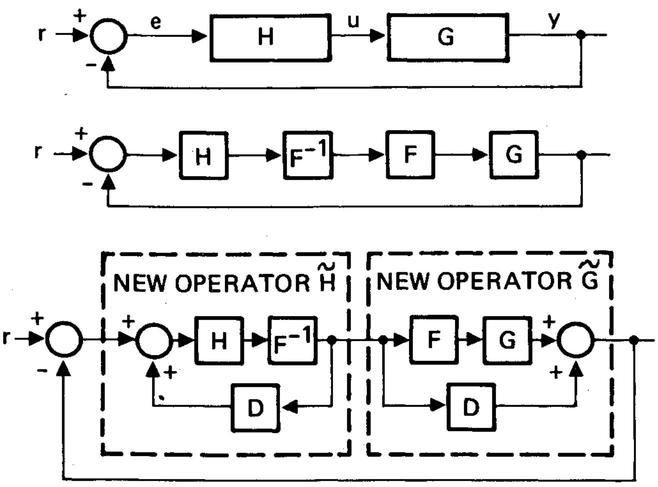

Fig. 2 Block diagram transformations for embedding.

realization of Z if

$$Z(s) = C(sI-A)^{-1}B$$

This is because $Z(s)$ is the transfer matrix relating the input vector u to the output vector y in a system having a state-space representation given by

$$\dot{x} = Ax + Bu \qquad y = Cx$$

Definition 3: The matrix triple $\{A,B,C\}$ is called a minimal realization of $Z(s)$ (with $Z(\infty) = 0$) if A is a square matrix of minimal dimension for which

$$Z(s) = C(sI-A)^{-1}B$$

When $Z(\infty)$ is nonzero but finite, then by way of notation we say that $\{A,B,C,Z(\infty)\}$ is a minimal realization of $Z(s)$ if $\{A,B,C\}$ is a minimal realization of $Z(s)$-$Z(\infty)$.

The motivation for these definitions and the associated terminology will become clear in the following development.

Conditions for Stability

For ease of understanding, and because they are presently of main interest, we will consider here only linear LSS control systems. However, the proofs given in the Appendix treat the controller and the plant as general operators, not necessarily linear, defined on a Hilbert function space. This permits extension of the theory to nonlinear systems.

Let the vector-valued feedback system S of Fig. 1 represent a control system providing active damping for a large space structure (LSS). The square transfer matrix $G(s)$ represents the structure and the square transfer matrix $H(s)$ the controller (which may include a reduced state estimator). The vector u represents the control forces and torques applied to the structure by distributed sensors. The vector-valued function $r(t)$ is the reference input that may be required for structural figure control (for active vibration damping $r(t) = 0$), and $e(t)$ is the return difference or error in the output measurements. We note that the application of positivity concepts requires $H(s)$ and $G(s)$ to be "square" transfer matrices. This implies that the number of inputs equals the number of outputs, which usually means an equal number of actuators and sensors. One can, of course, have more sensors than actuators and let an output y_i be a linear combination of sensor measurements. Note that the dimension of the matrices H and G is indicative of the number of inputs and outputs, but not of the order of the system model or the controller.

The main stability result will now be stated in the form of a theorem.

Theorem 1: If in the feedback system S of Fig. 1 $G(s)$ and $H(s)$ are square transfer matrices, then the system S is asymptotically stable in the input/output sense if at least one of the transfer matrices is strictly positive real and the other is positive real.

Remark 2: Theorem 1 is also true when $H(s)$ is in the feedback path of the system S of Fig. 1.

The proof of Theorem 1 is presented in the Appendix. Asymptotic stability applies only to those states of G which are completely observable/controllable.

Other statements and proofs of essentially the same theorem can be found in Refs. 6 and 18. The proof given in the Appendix differs from other proofs in its simplicity and complete generality, as already noted above. Stability based on positivity is extensively applied in Ref. 18 in the design of model reference adaptive control systems.

To require that the plant $G(s)$ be positive real (or strictly positive real) may sometimes be too restrictive for Theorem 1 to be useful. A technique called "embedding" is then applied. It essentially is a block diagram transformation of the original system into a completely equivalent system that embeds the original transfer matrices (or operators) into new and different transfer matrices. One then requires positivity of these new transfer matrices to establish stability according to Theorem 1. The block diagram transformation sequence is shown in Fig. 2 and the corresponding new transfer matrices are given by

$$\tilde{H} = (I - F^{-1}HD)^{-1}F^{-1}H \tag{1}$$

and

$$\tilde{G} = GF + D \tag{2}$$

where F and D are any well-behaved transfer matrices (well-behaved operators), including constant matrices. Judicious selection of F and D will make it possible to establish positivity of $\tilde{G}$ even though the plant G is not a positive operator. The controller H, however, must "pay" for this, since it must be very stable and positive to start with in order that the new $\tilde{H}$ will remain positive. To see this, set $F = I$ and let D be a constant, diagonal positive definite matrix. This clearly helps to improve the positivity of $\tilde{G}$ (think of the trace of a matrix) but it wraps a *positive* feedback loop around H, so that H had to be very stable for $\tilde{H}$ to be (strictly) positive real.

A good choice for the operators F and D is

$$F = I + j\omega Q \tag{3}$$

with Q diagonal and positive, and D a constant positive definite matrix. Other choices are allowed and possible and left to the ingenuity of the individual wanting to make use of Theorem 1. Embedding techniques become important in the application of Theorem 1 when actuator and sensor dynamics are included in the system design model.[19]

To apply Theorem 1 to establish stability of actively controlled LSS, we need a method to test when a square transfer matrix is (strictly) positive real. Direct application of Definitions 1 and 2 is easy for a single-input/single-output system (by plotting the transfer function as a function of ω in the complex plane), but becomes more tedious for transfer matrices (i.e., multi-input/multi-output systems). A better way is to make use of results due to Anderson.[14,15]

Theorem 2 (Ref. 14): Let $Z(s)$ be a square matrix of rational transfer functions such that $Z(\infty)$ is finite and Z has poles which lie in $\mathrm{Re}[s]<0$ or are simple on $\mathrm{Re}[s]=0$. Let $\{A,B,C,Z(\infty)\}$ be a minimal realization of Z. Then $Z(s)$ is positive real if and only if there exists a symmetric positive-definite matrix P and matrices W_0 and L such that

$$PA+A^TP=-LL^T \tag{4}$$

$$W_0^TW_0=Z(\infty)+Z^T(\infty) \tag{5}$$

$$C^T=PB+LW_0 \tag{6}$$

The fact that the theorem allows only simple (nonrepeated) poles on the $j\omega$ axis need not worry us, since any real structure has some damping, no matter how small, so that repeated modal frequencies are removed from the $j\omega$ axis by this limiting argument. Furthermore, examination of Anderson's proof reveals that it would allow completely undamped modal frequencies of multiplicity greater than one because the normal coordinate formulation of the structure allows strict diagonalization even with repeated eigenvalues. The direct sum method of modal subspaces applies then to each mode, regardless of whether the modal frequency is repeated or not.

Theorem 3 (Ref. 15): $Z(s)$ is "strictly" positive real if for a sufficiently small real $\sigma>0$, $\hat{Z}(s)\triangleq Z(s-\sigma)$ is positive real.

Remark 3: Theorem 3 is equivalent to modifying Eq. (4) to

$$PA+A^TP=-Q$$

where $Q=LL^T$ must be positive definite.

The important thing to note in Theorem 2 is that it permits us to determine the output matrix C of a differential system such that its transfer matrix

$$Z(s)=C(sI-A)^{-1}B \tag{7}$$

is positive real. For example, when $Z(\infty)=0$, Eq. (5) disappears; one solves the Liapunov equation of Eq. (4) for P after selecting some L and then computes $C^T=PB$ via Eq. (6). Note that the structure of the input influence matrix B of a physical system is usually fixed by nature and is not easily changed by the system designer. Note also that because L is arbitrary, there is, in general, no unique C that will cause $Z(s)$ in Eq. (7) to be positive real.

Application to the Design of LSS Controllers

The dynamics of a linear structure may be represented in state space description by

$$\dot{x}=Ax+Bu \qquad \text{structural dynamics} \tag{8}$$

$$y=Cx \qquad \text{output} \tag{9}$$

where the n-dimensional state vector ($n=2l$) is defined by

$$x\triangleq(\eta_1,\dot{\eta}_1,\eta_2,\dot{\eta}_2,\ldots,\eta_l,\dot{\eta}_l)^T \tag{10}$$

The variables η_i represent the structural deformations in normal modal coordinates; that is, the physical coordinates q of the finite mass elements are given by $q=\Phi\eta$, where Φ is the modal matrix. Equation (10) indicates that l modes are used in the structural model. Ultimately only a few of these modes will be actively controlled and the remainder classified as residual modes. It is very important to note, in what follows, that l may be arbitrarily large without affecting the results; that is, the stability criterion does not require modal truncation for the model.

With the state x as defined in Eq. (10), it is easy to show that

$$A=\begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_l \end{bmatrix} \tag{11}$$

with

$$A_j=\begin{bmatrix} 0 & 1 \\ -\omega_j^2 & -2\alpha_j \end{bmatrix}$$

where ω_j is the frequency of the jth mode and $\alpha_j=\omega_j\zeta_j$, with ζ_j the damping ratio of the jth mode. We assume here that the zero-frequency rigid body modes were removed from the model by either considering them ignorable coordinates, or by incorporating simple models of a rigid body controller.[1] The input influence matrix B is given by

$$B=\begin{bmatrix} 0 \\ B_1 \\ \hline 0 \\ B_2 \\ \hline \vdots \\ \hline 0 \\ B_l \end{bmatrix} \qquad B_j=[\phi_j(z_{a1}),\phi_j(z_{a2}),\ldots,\phi_j(z_{am})] \tag{12}$$

where z_{a1} to z_{am} are the locations of the m actuators and ϕ_j is the jth mode shape. For rate output only, we have

$$C=[0\ C_{r1}\,|\,0\ C_{r2}\,|\cdots|\,0\ C_{rl}] \tag{13}$$

and for position output only

$$C=[C_{P1}\ 0\,|\,C_{P2}\ 0\,|\cdots|\,C_{Pl}\ 0] \tag{14}$$

For combined position and rate output, C is modified accordingly. The submatrices C_{rj} and C_{pj} are m-dimensional column matrices containing the values of the jth mode shape at the m-sensor locations.

We now want to apply Theorem 2 to determine the output matrix C in Eq. (9) such that the transfer matrix $G(s)$ of the structure given by

$$G(s)=C(sI-A)^{-1}B \tag{15}$$

is positive real. Because the structural dynamics have been presented in a block diagonal canonical form, it is easy to show that Eq. (4) is satisfied for a symmetric positive definite P given by

$$P=\begin{bmatrix} P_1 & & & 0 \\ & P_2 & & \\ & & \ddots & \\ 0 & & & P_l \end{bmatrix} \qquad P_j=\begin{bmatrix} \omega_j^2 & 0 \\ 0 & 1 \end{bmatrix} \tag{16}$$

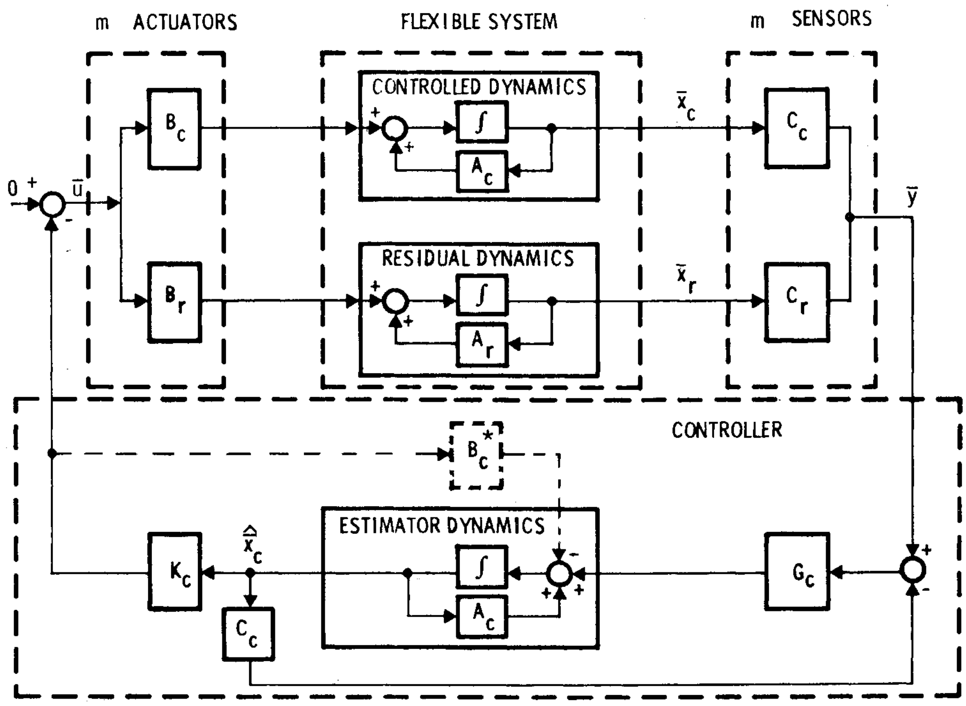

Fig. 3 System for active modal damping of a large flexible structure using positivity design techniques.

and block diagonal L with

$$L_j = \begin{bmatrix} 0 & 0 \\ 0 & 2\alpha_j^{1/2} \end{bmatrix} \tag{17}$$

Therefore, the transfer matrix $G(s)$ is positive real for $C^T = PB$ which, due to the peculiar structure of B, Eq. (9), yields $C^T = B$. We therefore have the following important result:

> The transfer matrix of a structure is positive real if collocated actuators and rate sensors are used. This result is totally independent of the numerical values of the modal frequencies and the mode shapes of the structure and does not require modal truncation.

We note that for this case we could not establish *strict* positive realness because LL^T is singular (not positive definite). We also note that when $\alpha = 0$, $L = 0$, which is perfectly acceptable, and Eq. (16) still holds. Other choices for L (which need not be square), and corresponding solutions for P satisfying Eq. (4), are clearly possible; some of these permit a combination of position and rate output. It turns out, however, that the permissible ratio of position-to-rate output that will ensure a positive real $G(s)$ is limited by structural damping (characterized here by α) which usually is very small. Thus, if position output is present or required, one should resort to the previously described embedding techniques.

The above given formulation of the problem assumed uncoupled modal damping. This is not necessary, and the obtained results hold also when the modes are coupled through damping. In this case it is better to make a different state assignment, i.e.,

$$x = \begin{bmatrix} \eta \\ \dot{\eta} \end{bmatrix}$$

The A matrix assumes, then, a block form of the form of the submatrix A_j and the results follow by considering the entire system (of arbitrarily large order) at once, instead of one mode at a time.

With $C^T = B$ (actuator collocated rate output), $G(s)$ is positive real and, by Theorem 1, stability of the closed-loop system S can be assured if the transfer matrix $H(s)$ of the controller is *strictly* positive real. We now consider two cases.

Case 1: $H(s)$ can easily be made strictly positive real if we select H *any positive definite* gain matrix (could be time-varying). This is a generalization of Canavin's results.[1,16]

Case 2: In order to use fewer sensors and actuators, one will be interested in using a dynamic compensator, i.e., an estimator/filter as a part of the controller. Figure 3 shows the assumed structure; the subscript c refers to the controlled modes. We can determine the gain matrix $G_c [n_c \times m$, where $n_c = 2\times$ (number of controlled modes) and $m =$ number of sensor/actuator pairs] so as to place the poles of the filter at desired locations (Luenberger observer), or we can determine G_c as the steady-state Kalman filter gain. All position and rate states are observable with a few judiciously placed rate sensors (perhaps only *one* sensor depending on multiplicity of eigenvalues) provided there are no zero-frequency modes present. We then apply Theorems 2 and 3 to determine the feedback gain matrix K_c (which is the output matrix of the filter) such that $H(s)$ is strictly positive real. To satisfy the modified Eq. (4) (see Remark 3) requires the solution of the Liapunov equation

$$P\Gamma + \Gamma^T P = -Q \tag{18}$$

for P, where $\Gamma = A_c - G_c C_c$ and Q may be any positive definite matrix. The choice of Q provides the needed design freedom to adjust performance. Then

$$K_c^T = PG_c \tag{19}$$

and a design assuring global stability of the system has been completed. This design is totally robust in stability since the

Table 1 First ten modal frequencies for simply supported flat plate

Mode number j	Mode shape[a] integers k	l	Modal frequency, Hz
1	1	1	3.24
2	1	2	8.10
3	2	1	8.10
4	2	2	12.96
5	3	1	16.20
6	1	3	16.20
7	2	3	21.00
8	3	2	21.00
9	4	1	27.5
10	1	4	27.5

[a] The mode shapes are obtained from the equation

$$\phi_j(x,y) = 0.449 \sin\left(\frac{k\pi x}{1.5}\right)\sin\left(\frac{l\pi y}{1.5}\right)$$

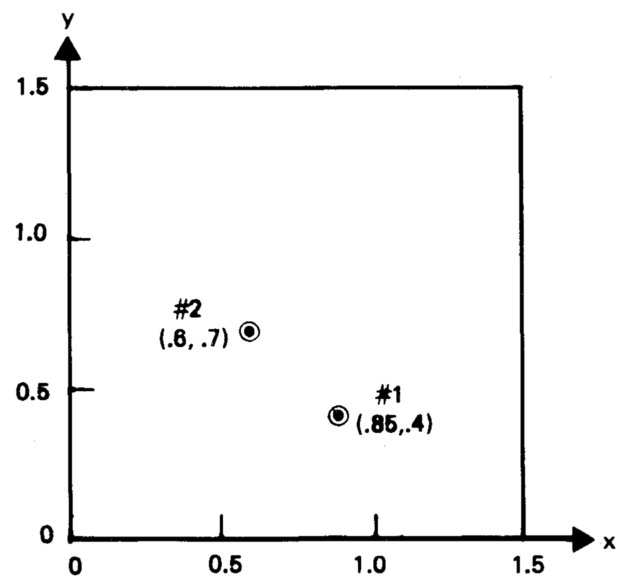

Fig. 4 Stations of collocated sensor and actuator pairs.

positivity of the transfer matrices remains intact even for large changes in the modal data and absolutely no assumption regarding modal truncation has to be made.

Other methods of design of the controller $H(s)$ subject to strict positivity are possible. One could, for example, determine K_c and G_c based on linear quadratic Gaussian (LQG) control, and then modify them, if necessary, to satisfy Eqs. (18) and (19). It is easy to show that the feedback gain matrix K_c is in either case of the familiar optimal regulator form, $K_c = R^{-1}B_c^T\tilde{P}$, but that the matrix $\tilde{P}$ is arrived at differently. Finding conditions under which the matrix $\tilde{P}$ associated with the positivity design technique equals the matrix $\tilde{P}$ computed for LQG control is an interesting topic for future research. In any event, it is important to note that design iterations to improve performance or assure global stability need only be performed on the low-order controller and no high-order "truth model" of the plant is required at all.

Example

For an example we consider a simply supported (i.e., hinged/pivoted at each edge) 1.5 × 1.5 m aluminum plate that is 1.5 mm thick. The first ten modal frequencies and a closed-form expression for the mode shapes are given in Table 1. We want to control the first three modes and since repeated modal frequencies are present, we require at least two sensor/actuator pairs to assure observability/controllability. We chose two point-force actuators and two rate sensors collocated as shown in Fig. 4.

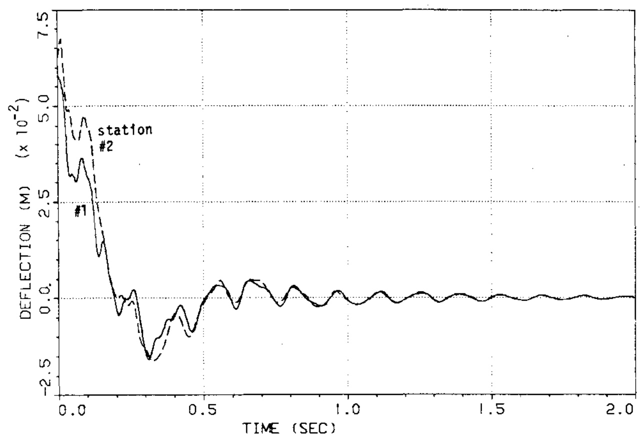

Fig. 5 Plate deflection at output stations.

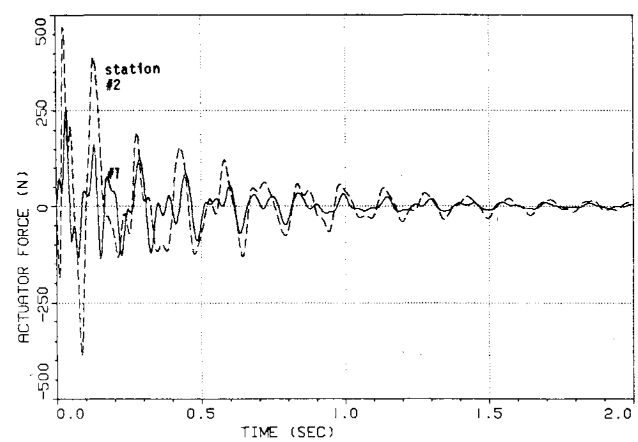

Fig. 6 Required actuator control force.

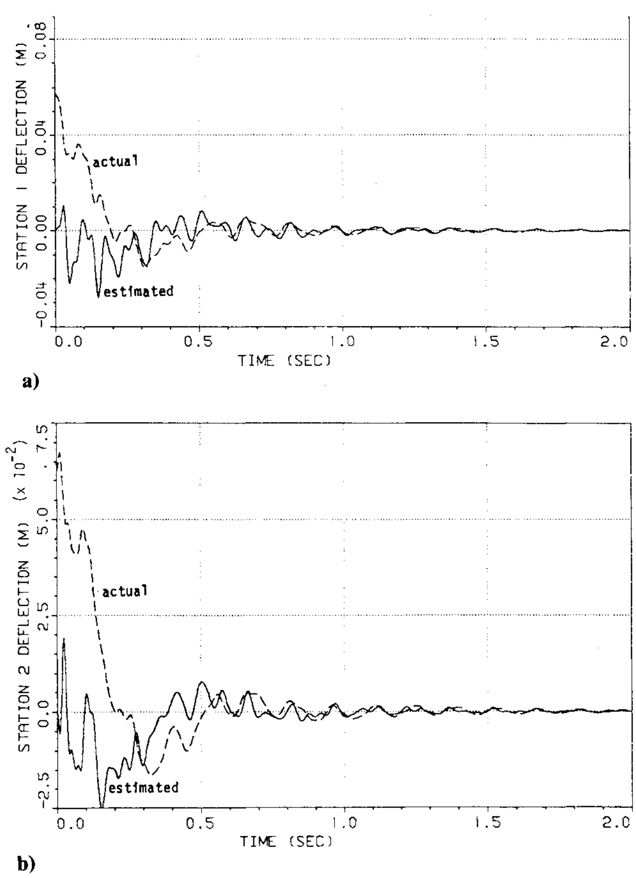

Fig. 7 Estimator predicted vs actual plate deflections: a) station 1, b) station 2.

Table 2 Filter/controller matrices

Optimal estimator gains

GC

−4.63E−01	−1.71E+00
2.76E+01	1.02E+02
−4.28E+00	−5.44E−01
1.84E+02	−4.02E+01
2.20E+00	−5.87E+00
−1.23E+02	2.06E+02

Feedback gains chosen using positivity

XK

−6.07E+02	2.64E+01	−1.23E+03	1.04E+02	1.10E+03	−3.51E+01
−2.25E+03	9.78E+01	8.97E+02	5.36E+01	−9.01E+02	1.72E+02

Estimator system matrices

AC

0.	1.00E+00	0.	0.	0.	0.
−4.14E+02	0.	0.	0.	0.	0.
0.	0.	0.	1.00E+00	0.	0.
0.	0.	−2.59E+03	0.	0.	0.
0.	0.	0.	0.	0.	1.00E+00
0.	0.	0.	0.	−2.59E+03	0.

CC

0.	3.26E−01	0.	4.37E−01	0.	−1.36E−01
0.	4.25E−01	0.	8.88E−02	0	2.62E−01

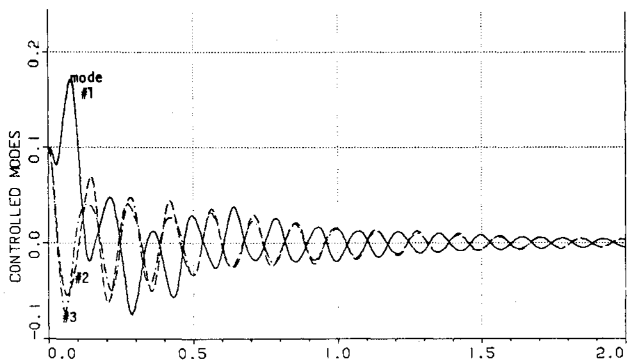

Fig. 8a Controlled modes response (3 modes).

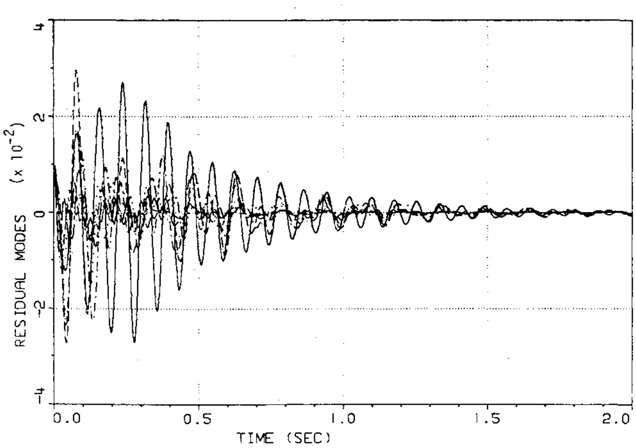

Fig. 8b Residual modes response (7 modes).

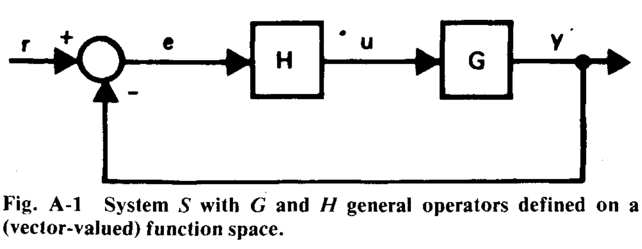

Fig. A-1 System S with G and H general operators defined on a (vector-valued) function space.

The previous results assure us that the plate transfer matrix will be positive real, regardless of exact values of modal frequencies and no modal truncation is required. We design a six-dimensional filter/controller $H(s)$ to control the first three modes as described for case 2. The design results for the filter $H(s)$ are given in Table 2. Plots of the closed-loop response of the first six modes of the plate, and various other quantities of interest, in response to an initial deformation, are shown in Figs. 5-8. As can be seen, the system is stable and the vibrations die out in about two seconds.

Conclusions

The described stability approach based on the positivity of operators promises to be a powerful technique for the design of stable and robust control systems for large space structures. The method needs no high-order truth models, needs to make no assumptions regarding modal truncation of the model, and is robust (in the stability sense) with respect to numerical values of the modal frequencies and associated mode shapes.

Appendix

Proof of Theorem 1

The intent of this Appendix is to prove Theorem 1 in general form so that it becomes apparent that the underlying theory is not restricted to linear, time-invariant differential systems. Many of the results in this Appendix are taken directly from Iwens.[12]

First some notation and definitions are required.

Notation: $\mathcal{H}$ is a Hilbert space of real-valued functions (not necessarily scalar valued) defined on $[0,\infty)$ with scalar product $\langle \cdot,\cdot \rangle$ and corresponding norm $\|\cdot\|$.

Notation: For a square matrix, the notation $A \geq 0$ means A is nonnegative definite; $A>0$ means A is positive definite; if A is self-adjoint, $A \geq \delta > 0$ means A is positive definite and its smallest eigenvalue is bounded away from zero by at least δ.

Definition A-1—Truncation: Let $f(t)$ be a real-valued function (not necessarily scalar valued) defined on $[0,\infty)$. Define the truncation of $f(t)$ at $t=T$ as

$$f_T(t) = f(t) \qquad 0 \leq t \leq T$$

$$= 0 \qquad t > T$$

Definition A-2—Extended Hilbert Space: $\mathcal{H}_e$ is an extension of $\mathcal{H}$ such that $f \in \mathcal{H}_e$ if and only if $f_T \in \mathcal{H}$. Note that $\mathcal{H} \subset \mathcal{H}_e$.

Definition A-3—Causality: The operator H, $\mathcal{H}_e \to \mathcal{H}_e$, is causal if and only if

$$Hf_T = (Hf)_T \quad \text{on} \quad [0, T]$$

This means that any operator describing the input-output relationship of a nonanticipative system (all physical systems) is causal.

Theorem A-1: In the general feedback system S of Fig. A-1 let F and H be causal operators which map $\mathcal{H}_e \to \mathcal{H}_e$ and $\mathcal{H} \to \mathcal{H}$ and let the input $r(t) \in \mathcal{H}$. Then $e(t) \in \mathcal{H}$ if there exist constants $\delta > 0$, $\beta \geq 0$, or alternatively, $\delta \geq 0$, $\beta > 0$ such that for all $f(t) \in \mathcal{H}_e$ and all $T \geq 0$ the following positivity conditions are satisfied:

$$\langle f_T, Hf_T \rangle \geq \beta \|f_T\|^2 \tag{A1}$$

$$\langle f_T, Gf_T \rangle \geq \delta \|f_T\|^2 \tag{A2}$$

Proof: From Fig. A-1 the system equations are

$$e = r - Gu \qquad u = He$$

Truncating at $t = T$, taking the scalar product with u_T on both sides and invoking causality yields

$$\langle u_T, e_T \rangle = \langle u_T, r_T \rangle - \langle u_T, Gu_T \rangle$$

By Eq. (A2) with $\delta > 0$ this reduces to

$$\langle u_T, e_T \rangle \leq \langle u_T, r_T \rangle - \delta \|u_T\|^2$$

Add the quantity $\gamma \|u_T\|^2$ on both sides, where γ is a positive constant satisfying $0 < \gamma < \delta$,

$$\langle u_T, e_T \rangle + \gamma \|u_T\|^2 \leq \langle u_T, r_T \rangle - (\delta - \gamma) \|u_T\|^2$$

and after completing the square on the right-hand side, one obtains

$$\langle u_T, e_T \rangle + \gamma \|u_T\|^2$$

$$\leq - \|\sqrt{\delta - \gamma} u_T - r_T / 2(\delta - \gamma)^{1/2}\|^2 + \|r_T\|^2 / 4(\delta - \gamma)$$

Substituting on the left $u = He$, invoking causality, and further strengthening the inequality by deleting the negative term on the right-hand side, yields

$$\langle e_T, He_T \rangle + \gamma \|u_T\|^2 \leq \|r_T\|^2 / 4(\delta - \gamma)$$

and using Eq. (A1) with $\beta = 0$ and that $r \in \mathcal{H}$,

$$\|u_T\|^2 \leq \|r\|^2 / 4\gamma(\delta - \gamma) < \infty$$

Since γ is arbitrary as long as $0 < \gamma < \delta$, set $\gamma = \delta/2$. This choice of γ minimizes the right-hand side of the inequality yielding the tightest bound. Since the inequality must hold for all T, no matter how large, and since the right-hand side is independent of T, it follows that

$$\|u\|^2 \leq \|r\|^2 / \delta^2 < \infty$$

and $u \in \mathcal{H}$. But

$$u \in \mathcal{H} \Rightarrow \quad y = Gu \in \mathcal{H} \Rightarrow \quad e \in \mathcal{H}$$

The proof of the theorem when H is strictly positive ($\beta > 0$) and G only semipositive ($\delta \geq 0$) proceeds similarly. Using $\beta > 0$ and $\delta \geq 0$ one arrives at

$$\langle u_T, e_T \rangle \leq \langle u_T, r_T \rangle$$

Subtracting $\gamma \|u_T\|^2$, $\gamma > 0$, on both sides, using Eq. (A1) and completing the square on the right-hand side yields

$$\beta \|e_T\|^2 - \gamma \|u_T\|^2 \leq \|r_T\|^2 / 4\gamma$$

Successively strengthening the inequality and using $r \in \mathcal{H}$ leads to

$$(\beta - \gamma \|H\|^2) \|e_T\|^2 \leq \|r\|^2 / 4\gamma < \infty$$

Since for any $\beta > 0$ we can choose a $\gamma > 0$ such that $(\beta - \gamma \|H\|^2) > 0$, and since the right-hand side is independent of T, it follows that $e \in \mathcal{H}$. Q.E.D.

Remark A-1: Note that in the proof the *lack* of strict positivity of one operator was overcome by "borrowing" from the strict positivity of the other operator.

Application of Theorem A-1 to Linear Differential Systems

Let $\mathcal{H}$ be the real, vector-valued Hilbert space $L_2(0,\infty)$. Then satisfaction of Theorem A-1 establishes L_2-stability of the system S (output in L_2 if input in L_2). Suppose that for all $u \in \mathcal{H}_e$ the operator G: $\mathcal{H}_e \to \mathcal{H}_e$ and $\mathcal{H} \to \mathcal{H}$ represents a linear differential system and G is defined by

$$Gu(t) = \int_0^\infty g(t-\tau) u(\tau) \mathrm{d}\tau$$

where $g(t)$ is a square impulse response matrix (which is zero for negative arguments) and its elements are Fourier transformable. Then the real quadratic form (star denotes adjoint)

$$\langle u_T, Gu_T \rangle = \langle u_T, \tfrac{1}{2}(G + G^*) u_T \rangle$$

$$= \int_0^\infty u_T^T(t) \left[\int_0^\infty \frac{1}{2} \left(g(t-\tau) + g^T(\tau - t) \right) u_T(\tau) \mathrm{d}\tau \right] \mathrm{d}t$$

By Parseval's theorem[17]

$$\langle u_T, Gu_T \rangle$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2} U_T^*(j\omega) \left(G(j\omega) + G^*(j\omega) \right) \cdot U_T(j\omega) \mathrm{d}\omega$$

and if $G(j\omega)$ is (strictly) positive real, i.e.,

$$\tfrac{1}{2} [G(j\omega) + G^*(j\omega)] \geq \delta \geq 0 \quad \text{for all } \omega$$

then

$$\langle u_T, Gu_T \rangle \geq \frac{\delta}{2\pi} \int_{-\infty}^{\infty} U_T^*(j\omega) U_T(j\omega) \mathrm{d}\omega$$

When a strictly positive real matrix tends to zero as $\omega \to \infty$, then in order to find a $\delta > 0$, the other operator in the loop is embedded with a feedforward $D = \delta I$, $\delta > 0$, sufficiently small, and the strict and semistrict operators trade places.

Applying Parseval's theorem again, it follows that the positivity condition

$$\langle u_T, Gu_T \rangle \geq \delta \|u_T\|^2$$

is satisfied if the matrix $G(j\omega)$ is (strictly) positive real. This is what we wanted to show, i.e., Theorem 1 is a special case of Theorem A-1. Satisfaction of Theorem 1 in the main text therefore establishes L_2-stability of the system S. For operators describing linear differential systems, L_2-stability implies asymptotic stability; the Schwarz inequality assures boundedness and the Riemann-Lebesgue lemma[17] assures convergence to zero. Observability/controllability assures state convergence.

Remark A-2: Application of the theory to nonlinear and/or time-varying operators requires a practical methodology to establish the positivity conditions in Theorem A-1 [Eqs. (A1) and (A2)] for these operators.

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